

6.5.4 Flow rule associated with von Mises Yield Function

$$-f(\sigma_{ij}) = J_2 - k^2 = 0$$

- The flow rule can be written (associative flow rule)

$$d\epsilon_{ij}^P = d\lambda \frac{\partial f}{\partial \sigma_{ij}} = d\lambda \frac{\partial J_2}{\partial \sigma_{ij}} = d\lambda S_{ij}' \quad - \text{eq ①}$$

Homework 6. problem 1 prove $\frac{\partial J_2}{\partial \sigma_{ij}} = S_{ij}'$

- $d\lambda \begin{cases} = 0 & \text{when } J_2 - k^2 < 0 \text{ or } J_2 - k^2 = 0 \text{ but } dJ_2 < 0 \\ > 0 & \text{when } J_2 - k^2 = 0 \text{ and } dJ_2 = 0 \end{cases}$

- Prandtl-Reuss equation

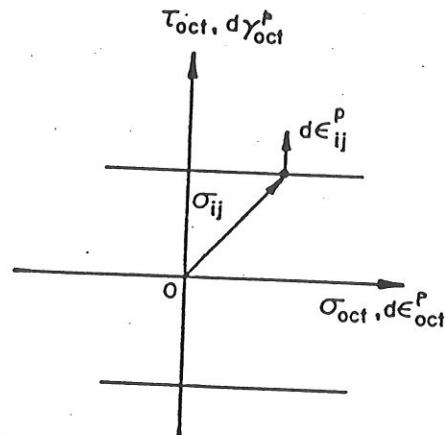
$$d\epsilon_{ij}^P = d\lambda S_{ij}' \Rightarrow d\lambda = \frac{d\epsilon_{ij}^P}{S_{ij}'}$$

$$\Rightarrow \frac{d\epsilon_{11}^P}{S_{11}'} = \frac{d\epsilon_{22}^P}{S_{22}'} = \frac{d\epsilon_{33}^P}{S_{33}'} = \frac{d\epsilon_{12}^P}{S_{12}'} = \frac{d\epsilon_{13}^P}{S_{13}'} = \frac{d\epsilon_{23}^P}{S_{23}'} = d\lambda$$

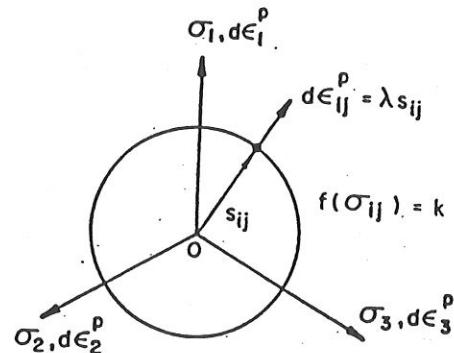
$$\left(\frac{d\gamma_{12}^P}{2\tau_{12}} = \frac{d\gamma_{13}^P}{2\tau_{13}} = \frac{d\gamma_{23}^P}{2\tau_{23}} = d\lambda \right)$$

- Prandtl first proposed Stress-strain relation for elastic and perfectly plastic material in 1924 under plane strain case
- Reuss extend Prandtl equation into 3-D stress-state and made above equation.
- Prandtl-Reuss equation means a small increment of plastic strain $d\epsilon_{ij}^P$ depends only on the current state of deviatoric stress S_{ij}' , not on the stress increment $d\sigma_{ij}$.
(Hydro-static pressure independent yield criterion)

- Graphic demonstration of associative flow rule for Von Mises Plasticity



(a) Hydrostatic plane



(b) Deviatoric plane

- Eg ① shows that there is no volumetric plastic deformation
 - $d\epsilon_{ii}^P = d\lambda S_{ii}' = 0$ ($S_{ii}' = \delta_{ii} = 0$ always)
 - from figure(a), $d\epsilon_{ii}^P$ is always normal to the hydrostatic pressure axis.
- If generalized Hooke's Law is applicable and material is homogeneous and isotropic, strain increment tensor can be written

$$\begin{aligned}
 \underline{d\epsilon_{ij}} &= \frac{1+\nu}{E} d\sigma_{ij} - \frac{\nu}{E} d\sigma_{kk} \delta_{ij} + d\lambda S_{ij}' \\
 \text{Total Strain} &\quad \xrightarrow{\text{Elastic strain.}} \quad \xrightarrow{\text{Plastic strain.}} \\
 &= \underline{\frac{d\sigma_{kk}}{9K} \delta_{ij}} + \underline{\frac{dS_{ii}'}{2G}} + d\lambda S_{ij}' \quad \left. \begin{array}{l} \text{--- eq (2)} \\ \text{Derivation.} \end{array} \right\} \\
 \circ \text{ Volumetric strain increment} & \quad \underline{d\epsilon_{ii}} = \frac{d\sigma_{kk}}{3K} \delta_{ii} = \frac{d\sigma_{kk}}{3K} (\delta_{ii} = 3) \\
 \circ \text{ deviatoric } & \quad \underline{d\epsilon_{ij}'} = \frac{1}{2G} dS_{ij}' + d\lambda S_{ij}'
 \end{aligned}$$

6.5.5 Flow rule associated with Tresca Yield Function

- assume $\sigma_1 > \sigma_2 > \sigma_3$, then yield function becomes

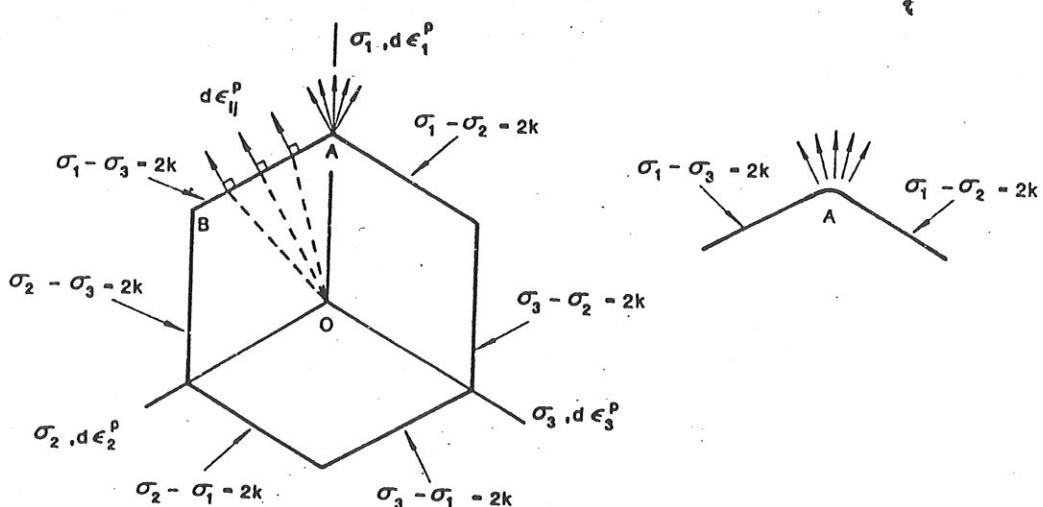
$$f = \sigma_1 - \sigma_3 - 2k = 0$$

- according to associative flow rule, the principal plastic strain increments have following relationships (within AB)

$$\begin{aligned} d\epsilon_1^p &= d\lambda \frac{\partial f}{\partial \sigma_1} = d\lambda \\ d\epsilon_2^p &= d\lambda \frac{\partial f}{\partial \sigma_2} = 0 \\ d\epsilon_3^p &= d\lambda \frac{\partial f}{\partial \sigma_3} = -d\lambda \end{aligned} \quad \left. \begin{array}{l} d\epsilon_1^p \\ d\epsilon_2^p \\ d\epsilon_3^p \end{array} \right\} = d\lambda \left. \begin{array}{l} 1 \\ 0 \\ -1 \end{array} \right\}$$

where $d\lambda > 0$

- flow rule can be applied same way for other five faces of yield surface.



- Assume $\sigma_1 > \sigma_2 = \sigma_3$ stress state.

- if we use $\sigma_1 - \sigma_3 = 2k$ then $\{d\epsilon_1^p, d\epsilon_2^p, d\epsilon_3^p\} = d\lambda \{1, 0, -1\}$
- if we use $\sigma_1 - \sigma_2 = 2k$ then $\{d\epsilon_1^p, d\epsilon_2^p, d\epsilon_3^p\} = d\lambda \{1, -1, 0\}$

- Use linear combination to compute plastic strain increment vector $d\epsilon_{ij}^p = \sum_{k=1}^n d\lambda_k \frac{\partial f_k}{\partial \sigma_{ij}}$

- plastic strain increment vector (tensor) $d\epsilon_j^P$ and stress vector (tensor) σ_{ij} can not make a one to one relationship

- Increment of plastic work (or rate of energy dissipation)

$$d\bar{W}_p = \sigma_{ij} d\epsilon_j^P = \sigma_1 d\epsilon_1^P + \sigma_2 d\epsilon_2^P + \sigma_3 d\epsilon_3^P$$

$$= 2k \max |\epsilon_i^P|$$

↑ max. shear stress

T 6.5-6 Flow rule associated with Mohr-Coulomb Yield Function

- Yield condition $\sigma_1 \frac{1 + \sin\phi}{2c \cos\phi} - \sigma_3 \frac{1 - \sin\phi}{2c \cos\phi} = 1$ or $m\sigma_1 - \sigma_3 = f_c'$ where $m = f_c'/f_c$ ($m > 1$)
- To obtain plastic strain increment $\{d\epsilon_1^P, d\epsilon_2^P, d\epsilon_3^P\}$ we have to consider three separate stress state
- Case 1) $\sigma_1 > \sigma_2 > \sigma_3$ and $d\lambda > 0$

$$d\epsilon_1^P = d\lambda \frac{df}{\sigma_1} = m d\lambda \quad d\epsilon_2^P = 0 \quad d\epsilon_3^P = -d\lambda$$

$$\{d\epsilon_1^P, d\epsilon_2^P, d\epsilon_3^P\} = d\lambda \{m, 0, -1\}$$

o Similar rule applies to five other faces of yield surface

o Volumetric strain increment

$$d\epsilon_v^P = d\epsilon_{ii}^P = d\epsilon_1^P + d\epsilon_2^P + d\epsilon_3^P = \underline{d\lambda(m-1)} \quad m = f_c'/f_c > 1$$

⇒ always predict volume dilation

o divide Volumetric strain increment into two parts.

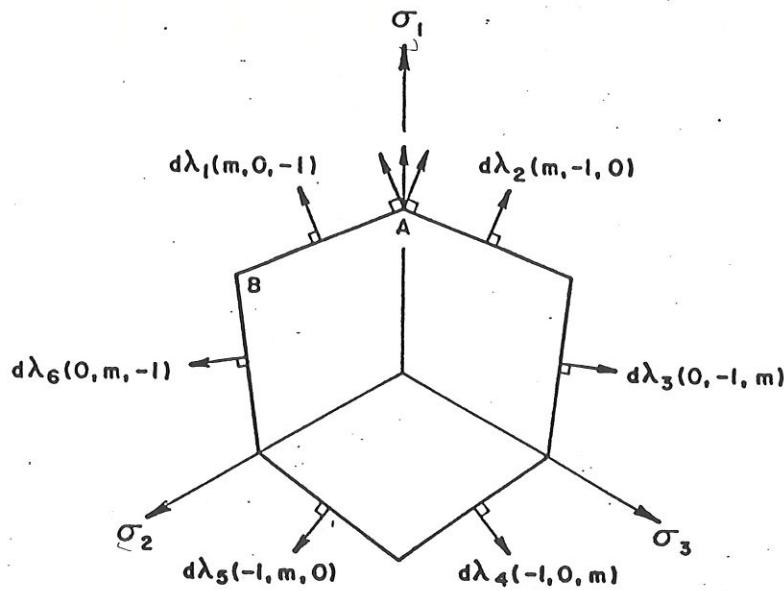
compressive part

tensile part

$$\sum |d\epsilon_c^P| = d\lambda$$

$$\sum d\epsilon_t^P = m d\lambda$$

$$\Rightarrow \frac{\sum d\epsilon_t^P}{\sum |d\epsilon_c^P|} = m \Rightarrow d\epsilon_v^P = \sum d\epsilon_t^P - \sum |d\epsilon_c^P|$$



- Plastic Work increment

$$dW_p = \sigma_1 d\epsilon_1^p + \sigma_2 d\epsilon_2^p + \sigma_3 d\epsilon_3^p = (\sigma_1 m - \sigma_3) d\lambda$$

from $m\sigma_1 - \sigma_3 = f_c'$, $\sum |d\epsilon_k^p| = d\lambda$, and $\sum d\epsilon_k^p = md\lambda$

$$dW_p = f_c' \sum |d\epsilon_k^p| = \frac{f_c'}{m} \sum d\epsilon_k^p$$

- (case 2) $\sigma_1 > \sigma_2 = \sigma_3$: Stress state lies on the edges of yield surface (ex. meridian A)

- From yield condition

$m\sigma_1 - \sigma_3 = f_c'$ and $m\sigma_1 - \sigma_2 = f_c'$ intersect along meridian A

- Linear combination rule from Tresca can be applied here

$$d\epsilon_{ij}^p = \sum_{k=1}^n d\lambda_k \frac{\partial f_k}{\partial \sigma_{ij}}$$

$$\begin{aligned} \Rightarrow \{d\epsilon_1^p, d\epsilon_2^p, d\epsilon_3^p\} &= d\lambda_1(m, 0, -1) + d\lambda_2(m, -1, 0) \\ &= \{(d\lambda_1 + d\lambda_2)m, -d\lambda_2, -d\lambda_1\} \end{aligned}$$

- The plastic strain increment vector lies between the directions of normals of two adjacent yield surfaces.

- The plastic volume change is obtained from

$$d\epsilon_v^p = m(d\lambda_1 + d\lambda_2) - (d\lambda_1 + d\lambda_2)$$

- Compressive part tensile part

$$\sum |d\epsilon_c^p| = d\lambda_1 + d\lambda_2 \quad \sum |d\epsilon_t^p| = m(d\lambda_1 + d\lambda_2)$$

$$\Rightarrow d\epsilon_v^p = \sum d\epsilon_t^p - \sum |d\epsilon_c^p|$$

- $d\epsilon_v^p > 0$ for $m > 1$: always predict volume dilation

$\text{eq}(*)$ $\frac{\sum d\epsilon_t^p}{\sum |d\epsilon_c^p|} = m$ and $d\epsilon_v^p = \sum d\epsilon_t^p - \sum |d\epsilon_c^p|$

- Plastic work increment

$$dW_p = (\sigma_i m - \sigma_3) d\lambda_1 + (\sigma_i m - \sigma_2) d\lambda_2$$

$\text{eq}(**) \quad = f'_c (d\lambda_1 + d\lambda_2) = f'_c \sum |d\epsilon_c^p|$

- Case 3) Stress state located at the apex of yield surface. where six surfaces are coincide.

- follow same rule for derivation.

- for $d\lambda_i$, $i = 1 \sim 6$

- $\sigma_1 = \sigma_2 = \sigma_3$

- $\text{eq}(*)$ and $\text{eq}(**)$ are still valid



linear combination

* William Warnke Formulation!

6.6 Elastic Perfectly Plastic Concrete Model

- Complete Stress-Strain relationship for concrete must be developed for three parts.
 - before yield \Rightarrow linear elastic stress-strain relationship
 - during plastic flow \Rightarrow Need to be discussed
 - after fracture \Rightarrow Stress-Strain model for fractured concrete (note p46~50) ^{w. f. Chen.} _{chap. 3}
- Need to determine two boundaries
 - Yield condition that defines beginning of plastic flow for concrete
 - Failure mode criterion that defines beginning of fracture (cracking or crushing type)
- Strain criterion is required for the mode of fracture of concrete under compression state
 \Rightarrow Simple conversion of criterion from stress (f_c') to crushing strain (ϵ_u)
- Stress-Strain relations for an elastic perfectly plastic material

$$d\epsilon_{ij} = \frac{dI_1}{9K} \delta_{ij} + \frac{dS'_{ij}}{2G} + d\lambda \frac{\partial f}{\partial \sigma_{ij}} \quad (\text{eq } ①)$$

$$d\lambda \begin{cases} = 0 & \text{when } f < k \text{ or } f = k \text{ but } df < 0 \\ > 0 & \text{when } f = k \text{ and } df = 0 \end{cases}$$

cop.

- Determination procedure of $d\lambda$

- combine eq① and $df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = 0$ (eq②)
- Assume stress state $(\sigma_{ij} + d\sigma_{ij})$ exists after the incremental change of stress $d\sigma_{ij}$.
- Assume it still satisfies the yield criterion f
 $f(\sigma_{ij} + d\sigma_{ij}) = f(\sigma_{ij}) + df = f(\sigma_{ij})$
because $df = 0$ if plastic flow occurs

- Solve eq② for dS'_{ij} , then the stress increment tensor can be determined

$$\begin{aligned} d\sigma_{ij} &= dS'_{ij} + \frac{1}{3} dI_1 d_{ij} \\ &= 2G d\epsilon_{ij} - 2G d\lambda \frac{\partial f}{\partial \sigma_{ij}} + \left(\frac{1}{3} - \frac{2}{9} \frac{G}{K} \right) dI_1 \delta_{ij} \end{aligned}$$

- Substitute $d\sigma_{ij}$ into eq③, then

$$(eq③) \quad 2G \frac{\partial f}{\partial \sigma_{ij}} d\epsilon_{ij} - 2G d\lambda \frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} + \left(\frac{1}{3} - \frac{2}{9} \frac{G}{K} \right) dI_1 \frac{\partial f}{\partial \sigma_{ij}} \delta_{ij} = 0$$

- To replace dI_1 , use eq① for $i=j \Rightarrow k$

$$d\epsilon_{kk} = \frac{dI_1}{9K} \delta_{kk} + \frac{dS'_{kk}}{2G} + d\lambda \frac{\partial f}{\partial \sigma_{ij}} \delta_{ij}$$

$$\Rightarrow dI_1 = 3K \left(d\epsilon_{kk} - d\lambda \frac{\partial f}{\partial \sigma_{ij}} \delta_{ij} \right) \quad (eq④)$$

- To compute $d\lambda$, substitute eq④ into eq③ and solve for $d\lambda$

$$d\lambda = \frac{\frac{\partial f}{\partial \sigma_{ij}} d\epsilon_{ij} + \frac{3K-2G}{6G} d\epsilon_{kk} \frac{\partial f}{\partial \sigma_{ij}} \delta_{ij}}{\frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} + \frac{3K-2G}{6G} \left(\frac{\partial f}{\partial \sigma_{ij}} \delta_{ij} \right)^2}$$

option 2

$d\lambda$ or $d\epsilon_{ij} \Rightarrow$ iteration ...

$d\epsilon_{ij} \leftarrow f_n.$

$\hookrightarrow d\lambda \text{ or } f_n.$

Homework 6 prob 2. derive eq ⑤

102

- If yield condition f is known for any particular material and strain increment $d\epsilon_{ij}$ are prescribed, $d\lambda$ can be defined uniquely.
- Substitute eq ④ into stress increment $d\sigma_{ij}$, then

$$d\sigma_{ij} = 2G d\epsilon'_{ij} + K d\epsilon_{kk} \delta_{ij} - d\lambda \left[\left(K - \frac{2}{3}G \right) \frac{\partial f}{\partial \sigma_{mn}} \delta_{mn} d\epsilon_{ij} + 2G \frac{\partial f}{\partial \sigma_{ij}} \right]$$
- Stress increment can be uniquely defined by above equation with strain increment, yield condition and $d\lambda$.
- If the current stress state σ_{ij} is known and the increment of strain $d\epsilon_{ij}$ are prescribed, eq ① can be used to determine corresponding stress increment $d\sigma_{ij}$.

— For generalized example, assume the yield condition is function of I_1 and J_2

$$f(\sigma_{ij}) = f(I_1, \sqrt{J_2}) = k$$

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial I_1} \frac{\partial I_1}{\partial \sigma_{ij}} + \frac{\partial f}{\partial J_2} \frac{\partial J_2}{\partial \sigma_{ij}}$$

$$= \frac{\partial f}{\partial I_1} \delta_{ij} + \frac{1}{2\sqrt{J_2}} \frac{\partial f}{\partial J_2} S'_{ij} \quad - \text{eq ⑥}$$

Homework 6 Prob 3

prove

$$\frac{\partial \sqrt{J_2}}{\partial \sigma_{ij}} = \frac{1}{2\sqrt{J_2}} S'_{ij}$$

$$\text{Hint: } S'_{ij} = (\sigma_{ij}' - \frac{1}{3} \sigma_{kk} \delta_{ij})$$

- Before substitute eq ⑥ into eq ⑤, compute $\frac{\partial f}{\partial \sigma_{mn}} \delta_{mn}$ in eq ⑤

$$\begin{aligned} \frac{\partial f}{\partial \sigma_{mn}} \delta_{mn} &= \frac{\partial f}{\partial I_1} \delta_{mn} \delta_{mn} + \frac{1}{2\sqrt{J_2}} \frac{\partial f}{\partial J_2} \overbrace{\delta_{mn} \delta_{mn}}^{\Rightarrow J_1 = 0} \\ &= 3 \frac{\partial f}{\partial I_1} \end{aligned} \quad - \text{eq ⑦}$$

$$df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = \left(\frac{\partial f}{\partial I_1} \delta_{ij} + \frac{1}{2\sqrt{J_2}} \frac{\partial f}{\partial \sqrt{J_2}} S'_{ij} \right) (2G dE'_{ij} + K d\varepsilon_{kk} \delta_{ij} - d\lambda [3K \frac{\partial f}{\partial I_1} \delta_{ij} + \frac{G}{\sqrt{J_2}} \frac{\partial f}{\partial \sqrt{J_2}} S'_{ij}])$$

103

- Substitute eq(6), (7) into eq(5), then

$$\begin{aligned} d\sigma_{ij} &= 2G dE'_{ij} + K d\varepsilon_{kk} \delta_{ij} \\ &\quad - d\lambda \left[(K - \frac{2}{3}G) 3 \frac{\partial f}{\partial I_1} \delta_{ij} + 2G \left(\frac{\partial f}{\partial I_1} \delta_{ij} + \frac{G}{\sqrt{J_2}} \frac{\partial f}{\partial \sqrt{J_2}} S'_{ij} \right) \right] \\ \text{eq(8)} \quad d\sigma_{ij} &= 2G dE'_{ij} + K d\varepsilon_{kk} \delta_{ij} - d\lambda \left[3K \frac{\partial f}{\partial I_1} \delta_{ij} + \frac{G}{\sqrt{J_2}} \frac{\partial f}{\partial \sqrt{J_2}} S'_{ij} \right] \end{aligned}$$

* Plastic deformation of
region

$$\hookrightarrow df = \frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = 0$$

- Compute $d\lambda$ with eq(8) (Homework 6, prob 4)

$$d\lambda = \frac{3K d\varepsilon_{kk} \left(\frac{\partial f}{\partial I_1} \right) + \frac{G}{\sqrt{J_2}} \frac{\partial f}{\partial \sqrt{J_2}} S'_{ij} dE'_{ij}}{9K \left(\frac{\partial f}{\partial I_1} \right)^2 + G \left(\frac{\partial f}{\partial \sqrt{J_2}} \right)^2} \quad - \text{eq(9)}$$

↓

6.6.1 Prandtl-Reuss Material (J₂ Theory)

- Prandtl-Reuss Material uses Von Mises with associative flow rule.

$$f = \sqrt{J_2} = K \quad (\text{or}) \quad f = \sqrt{J_2} - k = 0$$

- Use "Elastic and perfectly plastic material model"
- Compute $d\lambda$ from formulas in page 101 or 103 (eq(9))

$$d\lambda = \frac{S'_{ij}}{\sqrt{J_2}} dE'_{ij} \quad \text{deviatoric strain tensor.}$$

- total strain increment tensor is given by

$$d\varepsilon_{ij} = \frac{dS'_{ij}}{2G} + \frac{dI_1}{9K} \delta_{ij} + \frac{S'_{mn} dE'_{mn}}{2K^2} S'_{ij}$$

- Stress increment is computed from eq(8) and given by

$$d\sigma_{ij} = 2G dE'_{ij} + K d\varepsilon_{kk} \delta_{ij} - \frac{G S'_{mn} dE'_{mn}}{K^2} S'_{ij}$$

6.6.2 Drucker-Prager Material

$$f = \sqrt{J_2} + \alpha I_1 = k$$

α and $k > 0$ always

- the derivative of yield condition w.r.t stress

components is

(P 102 eq ⑥)

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial I_1} \frac{\partial I_1}{\partial \sigma_{ij}} + \frac{\partial f}{\partial \sqrt{J_2}} \frac{\partial \sqrt{J_2}}{\partial \sigma_{ij}}$$

$$= \alpha \delta_{ij} + \frac{S'_{ij}}{2\sqrt{J_2}}$$

- $d\lambda$ is given by Eq ⑨ in page 103

$$d\lambda = \frac{G}{\sqrt{J_2}} S'mn dE'mn + 3K\alpha dE'_{kk}$$

eq ⑩

- Strain increment tensor with associative flow rule is

$$d\varepsilon_{ij} = \frac{dS'_{ij}}{2G} + \frac{dI_1}{3K} \delta_{ij} + d\lambda \left(\frac{S'_{ij}}{2\sqrt{J_2}} + \alpha \delta_{ij} \right)$$

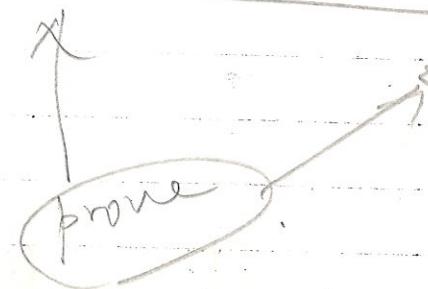
- Volumetric strain increment is

$$dE'_{kk} = \frac{dS'_{kk}}{2G} + \frac{dI_1}{3K} + d\lambda \left(\frac{S'_{kk}}{2\sqrt{J_2}} + 3\alpha \right)$$

$$= \frac{dI_1}{3K} + \frac{3\alpha d\lambda}{\text{Plastic part}}$$

elastic plastic part $\Rightarrow 3\alpha d\lambda > 0$ always

- Plastic deformation always increases volume of material.



- From Eq ⑧ and ⑨ in page 103, stress increment tensor for Drucker-Prager is given by

$$(d\sigma_{ij}) = 2G dE'_{ij} + K dE_{kk} \delta_{ij} - d\lambda \left[3K \frac{\partial f}{\partial J_1} S_{ij}' + \frac{G}{\sqrt{J_2}} \frac{\partial f}{\partial \sqrt{J_2}} S_{ij}' \right]$$

$$= 2G dE'_{ij} + K dE_{kk} \delta_{ij} - d\lambda \left[3K \alpha \delta_{ij} + \frac{G}{\sqrt{J_2}} S_{ij}' \right]$$

where $d\lambda = \frac{G}{\sqrt{J_2}} S_{mn}' dE_{mn} + 3K \alpha dE_{kk}$

- Elasto-plastic-constitutive (stiffness) tensor for finite element formulation is given by $d\sigma_{ij} = C_{ijmn} dE_{mn}$

$$d\sigma_{ij} = 2G \left[dE_{ij} - \frac{1}{3} dE_{kk} \delta_{ij} \right] + K dE_{kk} \delta_{ij}$$

$$- \frac{G}{\sqrt{J_2}} S_{mn}' dE_{mn} + 3K \alpha dE_{kk} \left[3K \alpha \delta_{ij} + \frac{G}{\sqrt{J_2}} S_{ij}' \right]$$

Therefore $C_{ijmn} = 2G \delta_{im} \delta_{jn} + \left(K - \frac{2}{3} G \right) \delta_{ij} \delta_{mn}$

$$\therefore \frac{\left(3K \alpha \delta_{ij} + \frac{G}{\sqrt{J_2}} S_{ij}' \right)}{G + 9K\alpha^2} \left(\frac{G}{\sqrt{J_2}} S_{mn}' + 3K \alpha S_{mn} \right)$$

6.6.3 Mohr - Coulomb Material

- Rewrite yield condition from page 90

$$f(I, \sqrt{J_2}, \theta) = I_1 \sin\phi + \frac{3(1-\sin\phi) \sin\theta + \sqrt{3}(3+\sin\phi) \cos\theta}{2} \sqrt{J_2}$$

- The angle of similarity

$$\cos 3\theta = \frac{3\sqrt{3}}{2} \frac{J_3}{J_2^{3/2}}$$

$$-3C \cos\phi = 0$$

- To obtain derivative of yield surface, we need

$$\frac{\partial \theta}{\partial J_2} \text{ and } \frac{\partial \theta}{\partial J_3}$$

$$\frac{\partial \cos 3\theta}{\partial J_2} = \frac{3\sqrt{3}}{2} \frac{\partial}{\partial J_2} \left(\frac{J_3}{J_2^{3/2}} \right)$$

$$\frac{\partial \cos 3\theta}{\partial \theta} \frac{\partial \theta}{\partial J_2} = - \frac{9\sqrt{3}}{4} \frac{J_3}{J_2^{5/2}}$$

$$-3 \sin 3\theta \frac{\partial \theta}{\partial J_2} = - \frac{9\sqrt{3}}{4} \frac{J_3}{J_2^{5/2}}$$

$$\frac{\partial \theta}{\partial J_2} = \frac{3\sqrt{3}}{4 \sin 3\theta} \frac{J_3}{J_2^{5/2}}$$

$$\frac{\partial \cos 3\theta}{\partial J_3} = \frac{3\sqrt{3}}{2} \frac{\partial}{\partial J_3} \left(\frac{J_3}{J_2^{3/2}} \right)$$

$$-3 \sin 3\theta \frac{\partial \theta}{\partial J_3} = \frac{3\sqrt{3}}{2} \frac{1}{J_2^{3/2}}$$

$$\frac{\partial \theta}{\partial J_3} = \frac{-\sqrt{3}}{2 \sin 3\theta} \frac{1}{J_2^{3/2}}$$

- Compute derivative of Mohr-Coulomb yield criterion

$$\frac{\partial f}{\partial \sigma_j} = \frac{\partial f}{\partial I_1} \frac{\partial I_1}{\partial \sigma_j} + \frac{\partial f}{\partial J_2} \frac{\partial J_2}{\partial \sigma_j} + \frac{\partial f}{\partial J_3} \frac{\partial J_3}{\partial \sigma_j}$$

$$\frac{\partial f}{\partial I_1} = \sin\phi$$

$$\frac{\partial f}{\partial J_2} = \frac{3(1-\sin\phi) \sin\theta + \sqrt{3}(3+\sin\phi) \cos\theta}{4\sqrt{J_2}}$$

$$+ \frac{3\sqrt{3} J_3 [3(1-\sin\phi) \cos\theta - \sqrt{3}(3+\sin\phi) \sin\theta]}{8 J_2^2 \sin 3\theta}$$

107

$$\frac{\partial f}{\partial J_3} = - \frac{\sqrt{3} [3(1 - \sin \phi) \cos \theta - \sqrt{3}(3 + \sin \phi) \sin \theta]}{4 J_2 \sin 30}$$

$$\frac{\partial I_1}{\partial \sigma_{ij}} = d_{ij}$$

$$\frac{\partial J_2}{\partial \sigma_{ij}} = S'_{ij}$$

$$\begin{aligned} \frac{\partial J_3}{\partial \sigma_{ij}} &= \frac{\partial}{\partial \sigma_{ij}} \left(\frac{1}{3} S'_{km} S'_{mn} S'_{ne} \right) \\ &= \frac{1}{3} \left[\frac{\partial}{\partial \sigma_{ij}} \left(\sigma_{em} - \frac{1}{3} \sigma_{kk} \delta_{em} \right) S'_{mn} S'_{ne} + S'_{em} \frac{\partial}{\partial \sigma_{ij}} \left(\sigma_{mn} - \frac{1}{3} \sigma_{kk} \delta_{mn} \right) S'_{ne} \right. \\ &\quad \left. + S'_{em} S'_{mn} \frac{\partial}{\partial \sigma_{ij}} \left(\sigma_{ne} - \frac{1}{3} \sigma_{kk} \delta_{ne} \right) \right] \\ &= \frac{1}{3} \left[\left(\delta_{ia} \delta_{jm} - \frac{1}{3} \delta_{em} \delta_{ij} \right) S'_{mn} S'_{ne} + \left(\delta_{im} \delta_{jn} - \frac{1}{3} \delta_{mn} \delta_{ij} \right) S'_{em} S'_{ne} \right. \\ &\quad \left. + \left(\delta_{in} \delta_{je} - \frac{1}{3} \delta_{ne} \delta_{ij} \right) S'_{em} S'_{mn} \right] \\ &= \frac{1}{3} \left[S'_{jn} S'_{ni} - \frac{1}{3} S'_{en} S'_{ne} \delta_{ij} + S'_{ei} S'_{je} - \frac{1}{3} S'_{em} S'_{me} \delta_{ij} \right. \\ &\quad \left. + S'_{jm} S'_{mi} - \frac{1}{3} S'_{em} S'_{me} \delta_{ij} \right] \\ &= \frac{1}{3} \left[S'_{jn} S'_{ni} + S'_{je} S'_{ei} + S'_{jm} S'_{mi} \right. \\ &\quad \left. - \frac{1}{3} (S'_{en} S'_{ne} + S'_{em} S'_{me} + S'_{em} S'_{me}) \delta_{ij} \right] \\ &= S'_{ik} S'_{kj} - \frac{2}{3} J_2 \delta_{ij} \end{aligned}$$

To derive stress and strain increment tensors
use following three equations.

$$d\lambda = \frac{\frac{\partial f}{\partial \sigma_{ij}} d\varepsilon_{ij} + \frac{3K - 2G}{6G} d\varepsilon_{kk} \frac{\partial f}{\partial \sigma_{ij}} \delta_{ij}}{\frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} + \frac{3K - 2G}{6G} \left(\frac{\partial f}{\partial \sigma_{ij}} \delta_{ij} \right)^2}$$

$$d\varepsilon_{ij} = \frac{dI}{9K} \delta_{ij} + \frac{ds'_{ij}}{2G} + d\lambda \frac{\partial f}{\partial \sigma_{ij}}$$

$$d\sigma_{ij} = 2G d\varepsilon'_{ij} + K d\varepsilon_{kk} \delta_{ij} - d\lambda \left[\left(K - \frac{2G}{3} \right) \frac{\partial f}{\partial \sigma_{mn}} \delta_{mn} \delta_{ij} + 2G \frac{\partial f}{\partial \sigma_{ij}} \right]$$

6.6.4. Willam-Warnke material

- Three parameter model with straight meridian

$$f(\sigma_m, \tau_m, \theta) = \frac{1}{\psi} \frac{\sigma_m}{f_c} + \frac{1}{P(\theta)} \frac{\tau_m}{f_c} - 1 = 0$$

where $\sigma_m = \frac{1}{3} I_1 = \frac{1}{3} (\sigma_1 + \sigma_2 + \sigma_3)$

$$\tau_m = \sqrt{\frac{2}{5} J_2} = \frac{1}{\sqrt{15}} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2}$$

$$P(\theta) = \frac{2p_c(p_c^2 - p_t^2) \cos \theta + p_c(2p_t - p_c)[4(p_c^2 - p_t^2) \cos^2 \theta + 5p_t^2 - 4p_t p_c]}{4(p_c^2 - p_t^2) \cos^2 \theta + (p_c - 2p_t)^2}$$

assume $P(\theta) = \frac{U(\theta)}{V(\theta)}$

- from angle of similarity $(\sigma_1 \geq \sigma_2 \geq \sigma_3)$

$$\cos \theta = \frac{2\sigma_1 - \sigma_2 - \sigma_3}{\sqrt{2} [(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]^{1/2}} = \frac{P}{g} \quad (\text{page 55})$$

- The gradient of yield condition

$$\frac{\partial f}{\partial \sigma_{ij}} = \frac{\partial f}{\partial \sigma_m} \frac{\partial \sigma_m}{\partial \sigma_{ij}} + \frac{\partial f}{\partial \tau_m} \frac{\partial \tau_m}{\partial \sigma_{ij}} + \frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial \sigma_{ij}}$$

$$1 \frac{3}{6} \quad \frac{\partial f}{\partial \sigma_m} \frac{\partial \sigma_m}{\partial \sigma_{ij}} = \frac{\partial f}{\partial \sigma_m} \frac{\partial \left(\frac{I_1}{3}\right)}{\partial \sigma_{ij}} = \frac{1}{\psi f'_c} \frac{1}{3} S_{ij}$$

$$2 \frac{3}{6} \quad \frac{\partial f}{\partial \tau_m} \frac{\partial \tau_m}{\partial \sigma_{ij}} = \frac{\partial f}{\partial \tau_m} \frac{\partial \left(\sqrt{\frac{2}{5} J_2}\right)}{\partial \sigma_{ij}} = \frac{1}{P(\theta) f'_c} \frac{S_{ij}'}{\sqrt{10} J_2}$$

$$= \frac{1}{P(\theta) f'_c} \frac{S_{ij}'}{5 \tau_m}$$

338

$$\frac{\partial f}{\partial \theta} \frac{\partial \theta}{\partial \sigma_{ij}} = \frac{\partial f}{\partial p} \frac{\partial p}{\partial \theta} \frac{\partial \theta}{\partial \sigma_{ij}}$$

$$\frac{\partial f}{\partial p} = -\frac{t_m}{f' c_p^2(\theta)} \frac{1}{p^2(\theta)}$$

$$\frac{\partial p}{\partial \theta} = \frac{\frac{\partial u}{\partial \theta} v - u \frac{\partial v}{\partial \theta}}{v^2}$$

$$\text{where } p(\theta) = \frac{u(\theta)}{v(\theta)}$$

$$\frac{\partial u}{\partial \theta} = 2\rho_c (\rho_t^2 - \rho_c^2) \sin \theta + \frac{4\rho_c (2\rho_t \rho_c) (\rho_t^2 - \rho_c^2) \cos \theta \sin \theta}{\sqrt{4(\rho_c^2 - \rho_t^2) \cos^2 \theta + 5\rho_t^2 - 4\rho_t \rho_c}}$$

$$\frac{\partial v}{\partial \theta} = 8(\rho_t^2 - \rho_c^2) \sin \theta \cos \theta$$

$$\frac{\partial \theta}{\partial \sigma_{ij}} = \frac{2(\cos^{-1}(p/\theta))}{2\sigma_{ij}} = \frac{1}{\sin \theta} \frac{\frac{\partial p}{\partial \sigma_{ij}} g - p \frac{\partial g}{\partial \sigma_{ij}}}{g^2}$$

$$\text{where } \frac{\partial p}{\partial \sigma_{ij}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\frac{\partial g}{\partial \sigma_{ij}} = \left(\frac{6}{5}\right)^{\frac{1}{2}} \frac{1}{\epsilon_m} S_{ij}'$$

$$= \frac{1}{3} \left(\frac{6}{5}\right)^{\frac{1}{2}} \frac{1}{\epsilon_m} \begin{bmatrix} 2\sigma_1 - \sigma_2 - \sigma_3 & 0 & 0 \\ 0 & 2\sigma_2 - \sigma_1 - \sigma_3 & 0 \\ 0 & 0 & 2\sigma_3 - \sigma_1 - \sigma_2 \end{bmatrix}$$

- after compute $\frac{\partial f}{\partial \sigma_{ij}}$

$$\text{use } d\lambda = \frac{\frac{\partial f}{\partial \sigma_{ij}} d\epsilon_{ij} + \frac{3k-24}{6G} d\epsilon_{kk} \frac{\partial f}{\partial \sigma_{ij}} d_{ij}}{\frac{\partial f}{\partial \sigma_{ij}} \frac{\partial f}{\partial \sigma_{ij}} + \frac{3k-24}{6G} \left(\frac{\partial f}{\partial \sigma_{ij}} d_{ij} \right)^2}$$

$$d\epsilon_{ij} = \frac{dI_i}{qK} d_{ij} + \frac{ds'_{ij}}{2G} + d\lambda \frac{\partial f}{\partial \sigma_{ij}}$$

$$d\sigma_{ij} = 2G d\epsilon_{ij} + K d\epsilon_{kk} d_{ij} - d\lambda \left[\left(k - \frac{2}{3} G \right) \frac{\partial f}{\partial \sigma_{mn}} d_{mn} d_{ij} - \frac{24}{G} \frac{\partial f}{\partial \sigma_{ij}} \right]$$

6.6.5 Elastic cracking formulation for concrete modeling

- Tension cut off criterion
- Material develops plastic deformation if the maximum principal stress exceeds yield stress.
⇒ Same as Rankine criterion
- Cover tensile regime formulation of elastic-plastic concrete modeling
- Yield criterion

$$f(\sigma_{ij}) = \sigma_i - \sigma_e = 0 \quad \text{with } \sigma_1 \geq \sigma_2 \geq \sigma_3$$

σ_e corresponds to uniaxial tensile strength of concrete
(f_t')

- follow elastic-perfectly plastic material formulation
- Inelastic deformations due to crack do not contribute to the elastic strain energy (dE_{ij}^e)
 $d\sigma_{ij} dE_{ij}^e = 0$
 $\Rightarrow d\sigma_{ij} \perp dE_{ij}^e \Rightarrow$ corresponds to flow rule of perfect plasticity.
- inelastic strain increment due to cracking is perpendicular to the fracture surface.

$$dE_{ij}^e = d\lambda \frac{\partial f}{\partial \sigma_{ij}}$$

- for convenience, define unit vector n_{ij} in the gradient direction as

$$n_{ij} = \frac{\frac{\partial f}{\partial \sigma_{ij}}}{\left| \frac{\partial f}{\partial \sigma_{ij}} \right|}$$

- for maximum principal stress tension cut off criterion

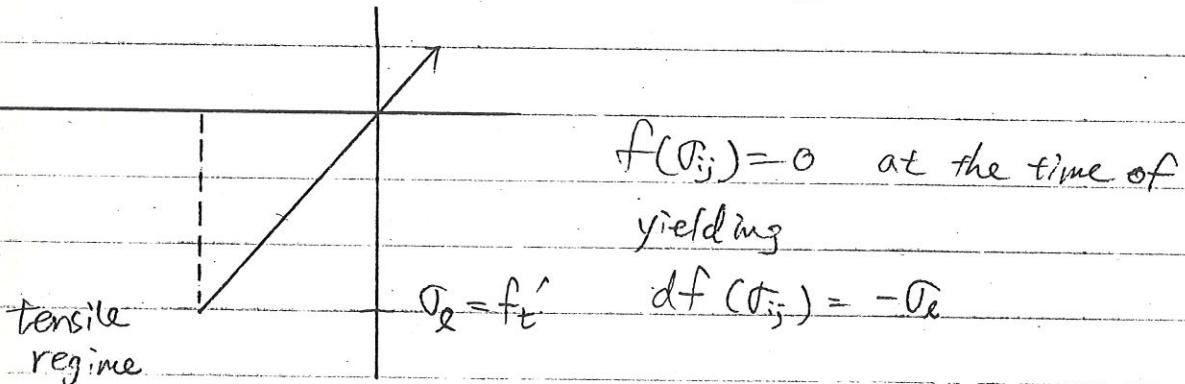
$$\frac{\partial f}{\partial \sigma_j} = \frac{\partial f}{\partial \sigma_1} \frac{\partial \sigma_1}{\partial \sigma_j}$$

$$n_{ij} = \frac{(\partial f / \partial \sigma_1) (\partial \sigma_1 / \partial \sigma_{ij})}{|\partial f / \partial \sigma_{ij}|} = \sigma_{ij} \delta_{ij}$$

where n_{ij} is the unit vector in the direction of σ_i

$$\{ n_{ij}^{(c)} \} = \{ 1, 0, 0, 0, 0, 0 \} \text{ and } |\partial f / \partial \sigma_{ij}| = 1$$

- for perfectly brittle behavior, the loading parameter $d\lambda$ is determined from softening condition



$$\text{Therefore } df = \frac{\partial f}{\partial \sigma_j} d\sigma_j = -\sigma_e$$

- Total strain increment

$$d\epsilon_{ij} = d\epsilon_{ij}^e + d\epsilon_{ij}^c$$

- elastic strain increment

$$d\sigma_{ij} = C_{ijk} e_k d\epsilon_{kj},$$

$$\text{where } C_{ijk} = (K - \frac{2}{3}G) \delta_{ij} \delta_{kk} + G (\delta_{ik} \delta_{j\ell} + \delta_{ik} \delta_{jk})$$

$$\frac{\partial f}{\partial \sigma_j} d\sigma_j = \frac{\partial f}{\partial \sigma_j} C_{ijk} (d\epsilon_{ij} - d\epsilon_{ij}^c) = -\sigma_e$$

$$\frac{\partial f}{\partial \sigma_{ij}} d\sigma_{ij} = \frac{\partial f}{\partial \sigma_{ij}} C_{jkl} (d\varepsilon_{ij} - d\lambda \frac{\partial f}{\partial \sigma_{ij}}) = -\sigma_e$$

$$\Rightarrow d\lambda = \frac{(\partial f / \partial \sigma_{ij}) C_{jkl} d\varepsilon_{kl} + \sigma_e}{(\partial f / \partial \sigma_{ij}) C_{jkl} \partial f / \partial \sigma_{ke}}$$